§6 IMP14  
Polygon Duals

## Point-Based Polygon Duals

Imagine a polygon in the Cartesian plane, and a point *P* somewhere in the plane. Imagine a point *T* traversing the polygon, and a unit vector based at *T* which points in the direction of travel of *T*, hence points along the edge which *T* is currently traversing. Refer to the unit vector as the *unit-sweep*. When *T* gets to a vertex, the unit-sweep vector rotates through the exterior angle of the polygon until it is in line with the next edge ready for the next traverse.

Now consider the vector *PT* (known as the *winding-vector*). While *T* is traversing an edge, *PT* changes its direction. While the unit-sweep vector is rotating at a vertex, the vector *PT* is stationary, but the angle of the unit-sweep vector is changing.

Suppose the polygon has *n* edges. Think of the movement of *T* as determined by a parameter which traverses a line segment. The line segment is divided into 2*n* consecutive segments of equal length. As a parameter point *t* traverses an odd-numbered segment, the point *T* is driven along the corresponding edge of the polygon, and when *t* traverses an even-numbered segment, the unit-sweep vector rotates through the exterior angle.

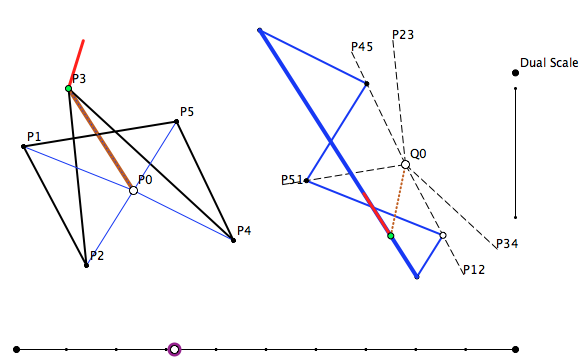
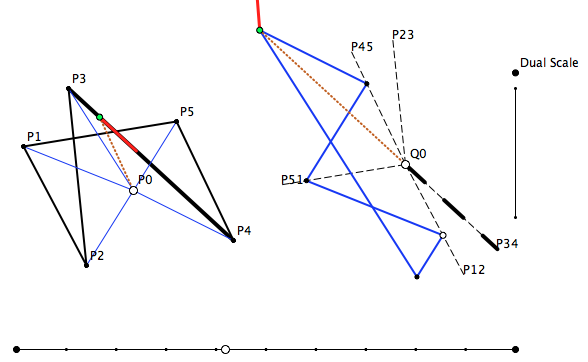
The motions of the unit-sweep vector and the winding-vector are thus determined by two sets of angles.

#### Point-Based Dual

The point-based dual of a polygon with respect to the point *P* is formed by interchanging the set of unit-sweep angles and the set of winding-vector angles, relative to some base point *Q*.

###### Dual of Dual

Convince yourself, and a friend, that the dual of the polygon with respect to the point *Q* which is itself the dual of a given polygon with respect to the point *P*  is always similar to the original. Note that the dual is only specified up to similarity.



Traversing an edge Traversing a vertex

See applet IMP Dual Polygons for examples.

#### Winding Number

As a point *T* traverses the edges of a polygon, the vector *PT* rotates about *P*. The *winding number* of the polygon with respect to *P* is the number of full revolutions made by the vector *PT* as *T* traverses the polygon. Note that if the point *T* traverses the polygon in the opposite direction, the winding number would be multiplied by –1. Consequently polygons are always considered to be oriented, so as to give a non-negative winding number.

###### Winding

Why is the winding number always an integer?

Is there necessarily a point *P* for which the winding number is 0?

Let 0 < *v* < *w* = winding number of a polygon with respect to the point *P*. Is there a point *Q* for which the winding number of the same polygon with respect to that point is *v*?

#### Sweep Number

As a point *T* traverses the edges of a polygon, the unit-sweep vector rotates about *T*. The *sweep number* of the polygon is the number of full revolutions made by the unit-sweep as *T* traverses the polygon.

###### Winding & Sweeping

How are the winding number with respect to a point *P* and the sweep number of a polygon related to the winding number of the *P* point-dual of the polygon and the sweep number of that dual?

### Swept-Regions and Swept-Numbers

Given a polygon and a point *P* in the plane, let *T* traverse the polygon. The *swept-number* of *P* with respect the polygon is the number of times that the line *PT* passes through a vertex or along an edge of the polygon. The swept-numbers of points partition the plane into *iso-* *swept-regions*

###### Winding & Sweeping Challenge

Given a polygon, find a way to locate the iso-swept-regions

## Projective Duals

The projective dual of a point is a line, and the projective dual of a line is a point. Briefly, present any point [*a*, *b*] in the *x*-*y*-plane as [*a*, *b*, 1] which is a line through the origin in three space meeting the plane *z* = 1 in the original point. The dual of this point is the line *ax* + *by* + 1 = 0. Similarly, given the line *ax* + *by* + *c* = 0 in the plane, it is equivalent to the line *ax*/*c* + *by*/*c* + 1 = 0 (as long as *c* ≠ 0), so its dual is the point [*a*/*c*, *b*/*c*, 1] on the plane *z* = 1, or [*a*/*c*, *b*/*c*] in the original *x-y-*plane. For more details see extra notes.

###### Dual Challenge

What is the connection between the point-based dual of a polygon with respect to a point *P* and the projective dual of the polygon?

The Applet Polygon Duals may be of assistance.

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## Notes on Projective Duals

The projective dual has a geometric interpretation as well as an algebraic procedure, but first it is necessary to embed the Cartesian plane in the projective plane, by adjoining a line at infinity made up of points, each of which corresponds to a family of parallel lines in the plane.

### The Projective Plane

Geometrically, imagine your eye (as a point) looking down on a plane in which lies the polygon *P*. The vertices of *P* correspond to rays of light, or lines through your eye. A line in the plane corresponds to all the rays through your eye to points on the line, which is a plane through your eye. Treat your eye as the origin of a vector space. Lines through your eye are ‘Ppoints’ and planes through your eye are ‘Llines’. The repeated letter is a reminder of where the object sits!

Algebraically this can be achieved by converting Cartesian coordinates [*a*, *b*] for a point into homogeneous coordinates {λ[*a, b*, 1] : λ in *R*} to describe the line through the origin. Since attention is on the lines as a direction rather than as made up of vectors, it is sufficient to use a single representative to refer to it. So the homogeneous coordinates [*a*, *b*, *c*] are equivalent to [μ*a*, μ*b*, μ*c*] for any non-zero μ.

#### Changing Planes of Graphs of Functions

A different projection of a configuration in the Cartesian plane can be obtained by forming the projective coordinates and then choosing a different plane (other than the plane *z* = 0) on which to project. For example, to project onto the plane *y* = 0, divide homogeneous coordinates by the *y*-coordinate and then delete the *y*-coordinate;

#### Result PP1

To project onto the plane α*x* + β*y* + γ*z* = 1, scale the homogeneous coordinates of a point so that the linear combination is 1, then eliminate any one of the coordinates (since it can be recovered from others);

###### Projecting a sine-wave

Find the function being plotted on each of the planes *y* = 0, *x* = 0, and *x* + *y* + *z* = 1 of the function [*x*, sin(*x*), 0] on the plane *z* = 0.

A line in the Cartesian plane corresponds to a plane through the eye. The equation of a plane through the eye is achieved by homogenising the equation of the line in the plane. Thus *ax* + *by* + *c* = 0 becomes the plane *ax* + *by* + *cz* = 0.

#### Result PD1

The line *a(x* – *p*) + *b*(*y* – *q*) + *c* = 0 in the Cartesian plane passes through the point [*p, q*] and has projective label [*a*, *b*, *ap* + *bq* + *c*].

The line through the point [*p,* *q*] in the Cartesian plane with slope *m* has the Cartesian equation *mx* – *y* + *q* – *mp* and so has projective label [*m*, –1, *q* – *mp*].

### Projective Dual

The construction is very simple: since points and lines both have triples of real numbers as their homogeneous (equivalence class of) coordinates, simply interchange points and lines. This means declaring the homogeneous coordinates of a point to be the homogeneous coordinates of a Dual-Point, namely a LLine and the coefficients of a line to be the coordinates of a Ppoint. Note, Cinderella, a dynamic geometry programme, actually uses homogeneous coordinates, converting these to Cartesian upon request, so the duality is easily achieved: simple tell the programme whether a triple is to be considered as a point or a line.

What might make a difference is where the origin is. The applet translates the polygon so that the given point *P0* is the origin, then interchanges points and lines, then translates the origin to *R0*.

#### Construction

Interchange points and lines

### Results

There are a number of questions which arise, especially when you want to present polygons and duals in dynamic geometry software.

#### Result PD1

The line through two points in homogeneous coordinates converts to the point of intersection of the lines dual to the two points.

The point of intersection of two lines in homogeneous coordinates converts to the line joining the points dual to the two lines.

#### Result PD2

To enable the user to move polygons around easily, it is helpful to present a polygon relative to a point, so that the polygon can be translated easily in the plane, and likewise its dual.

Given a polygon in the plane, presented in terms of a list of homogeneous coordinates of its vertices, relate the homogeneous coordinates of points relative to some displayed and moveable point *P*.

Present the vertices of the dual polygon relative to the point *Q*.

Present a line in homogeneous coordinates relative to a point *P*.

Present the dual point .

The cross-product of two homogenous triples is the homogeneous triple for the intersection of the corresponding lines, or the line joining the corresponding points. Think geometrically.

The cross product of the translation of two points is the translation of the cross product by the cross product of their difference with the translation.

## NOTES

Projective duals: p is incident with line L iff pL=0 as inner product (L written as column vectors

Line through p and q is p X q; p, q, r are collinear when det(p, q, r) = 0

L meets M at point L X M; L, M, N are coincident when det = 0

Polarity: p perp q rel to A if pAq(T)=0 where A is symmetric. Aq(T) is a line thru p and q is a point on line thru p

So, pole-polar duality wrt a conic is vector-space duality; the symmetric matrix A corresponds to a conic, and projectively, the plane perpendicular to the vector *p* is presented as the polar of the pole represented by *p*.

Appolonius knew about polarities wrt a conic via the tangent construction.